JOURNAL OF ALGEBRA 21, 422–427 (1972)

A Note on the Tensor Product of Algebras

R. HART

School of Mathematics, University of Leeds, Leeds, England

Communicated by A. W. Goldie

Received November, 1970

1. We suppose that rings have unit elements; "Noetherian" means that the maximum condition holds for left and right ideals. Let F be an algebraically closed field; we consider F-algebras A and B which are hereditary integral domains satisfying certain restrictions. It is shown that the global dimension of $A \otimes B$ is $\leqslant 2$ if the tensor product of the quotient division rings of A and B is a hereditary ring (this condition is not always satisfied). If F has characteristic zero and we denote by D_1 the quotient division ring of the (hereditary) algebra $A_1 = F[x, y](xy - yx = 1)$, then a necessary and sufficient condition on a division F-algebra D can be given for $D \otimes D_1$, $D \otimes A_1$ to be hereditary. The condition for $D \otimes A_1$ is given in Section 2; the other results follow the general Theorem 3.1, on which they depend.

2. We recall some notation and results from [1]. Let R be any Noetherian prime ring with quotient ring Q. When N is a left R-submodule of Q generated by finitely many units of Q, define N^* by

$$N^* = \{q \in Q \mid Nq \subseteq R\}.$$

Then N^* is a finitely generated right R-submodule of Q. (Similarly, if N is a right module, N^* is a left module). We have $N^{**} \supseteq N$ and $N^{***} = N^*$. Notice that if $N \subseteq N_1$, $N^* \supseteq N_1{}^*$. If N is generated by elements n_i, then N^* is a finite intersection of cyclic R-submodules $n_i^{-1}R$ of Q. Also N^* is naturally isomorphic to $\hom_R(N, R)$. From these facts it follows that if N is projective, then N^* is projective and $N = N^{**}$. Further, for any N, N^* can be embedded in a free module G in such a way that G/N^* is isomorphic to a submodule of a free module.

LEMMA 2.1 (see [1]). *Let R be a Noetherian integral domain, $\mathscr{S}$ a subset of R with respect to which R has a right and left quotient ring $R_{\mathscr{S}}$. Suppose $R_{\mathscr{S}}$ is a hereditary ring, and let $I \subseteq J$ be left ideals of R such that $R_{\mathscr{S}}I \neq R_{\mathscr{S}}J$. Then $I^* \neq J^*$.*

Proof. $R_{\mathscr{S}}I$, $R_{\mathscr{S}}J$ are distinct projective left ideals in $R_{\mathscr{S}}$. Hence $(R_{\mathscr{S}}I)^* \neq (R_{\mathscr{S}}J)^*$. In other words, there is an element q in the quotient ring of R such that $R_{\mathscr{S}}Iq \subseteq R$, $RAJq \nsubseteq R$. In particular, $Iq \subseteq R_{\mathscr{S}}$, $Jq \nsubseteq R_{\mathscr{S}}$. Now I is finitely generated; so by multiplying by a suitable element of $\mathscr{S}$, we may suppose $Iq \subseteq R$, $Jq \nsubseteq R$. Thus $I^* \neq J^*$.

LEMMA 2.2. *Let F be a field of characteristic zero, A an F-algebra which is a Noetherian integral domain with quotient division ring D. Let D_1 be the quotient division ring of A_1. Then*

$$\text{gl dim } D \otimes A_1 \leqslant 2, \qquad \text{gl dim } A \otimes A_1 \leqslant 2 + \text{gl dim } A,$$
$$\text{gl dim } A \otimes D_1 \leqslant 2 + \text{gl dim } A$$

Proof. Let S be any ring of global dimension n, $d : S \to S$ a derivation. Let R be the ring $S[x]$, where $xs - xs = ds$ for all $s \in S$. When $d = 0$, it is well known that gl dim $R \leqslant n + 1$. However, the proof of this fact on p. 45 of [2] works just as well if $d \neq 0$. gl dim $D \otimes A_1 \leqslant 2$ now follows by taking first $S = D$, $R = S[y_1]$, $d = 0$, and then $S = D \otimes F[y_1]$, $R = S[x_1]$, $dy_1 = 1$. Similarly gl dim $A \otimes A_1 \leqslant 2 +$ gl dim A follows by taking first $S = A$, $R = S[y_1]$, $d = 0$ and then $S = A \otimes F[y_1]$, $R = S[x_1]$, $dy_1 = 1$. Similarly gl dim $A \otimes A_1 \leqslant 2 +$ gl dim A follows by taking first $S = A$, $R = S[y_1]$, $d = 0$ and then $S = A \otimes F[y_1]$, $R = S[x_1]$, $dy_1 = 1$.

Now $A \otimes D_1$ is a partial quotient ring of the Noetherian ring $A \otimes A_1$. Therefore, $A \otimes D_1$ is Noetherian; so if I is a left ideal in $A \otimes D_1$, I is finitely generated. Hence we may suppose $I = \sum (A \otimes D_1) u_i$, where $u_i \in A \otimes A_1$. If $J = \sum (A \otimes A_1) u_i$, $I \cong D_1 \otimes_{A_1} J$. Therefore, under tensoring with D_1, a projective resolution of the $A \otimes A_1$-module J becomes a projective resolution of I as an $A \otimes D_1$-module. Thus gl dim $A \otimes D_1 \leqslant 2 +$ gl dim A.

THEOREM 2.3. *Let F be a field of characteristic zero, D a division F-algebra such that there is no isomorphism $A_1 \to D_n$ for any n. Then $R = D \otimes A_1$ is a hereditary ring.*

Proof. Each element of R is a polynomial in $1 \otimes x$, $1 \otimes y$ with coefficients in D. It follows that R is a Noetherian integral domain. Suppose $I \subsetneqq J$ are left ideals in R with J/I cyclic. Suppose $J = I + R\alpha$. If there were nonzero elements a in $D \otimes F[x]$ and b in $D \otimes F[y]$ such that $a\alpha \in I$, $b\alpha \in I$, then we would have $J/I \cong R/N$, where N is some left ideal in R containing a and b. Then R/N would be finite-dimensional over D, say of dimension n. Therefore, there would be a nonzero homomorphism $A_1 \to D_n$; in fact, an isomorphism, because A_1 is simple. This is not allowed. Let $\mathscr{S}$ be the set of nonzero elements in $D \otimes F[y]$. R has a quotient ring $R_{\mathscr{S}} = \Phi[x]$, where Φ is the quotient division ring of $D \otimes F[y]$, and for $\phi \in \Phi$, $x\phi - \phi x = \partial\phi/\partial y$. Similarly, when $\mathscr{S}$ is the set of nonzero elements in $D \otimes F[x]$. Therefore, for one or other of these multiplicatively closed sets (say, the first), we have $R_{\mathscr{S}}I \neq R_{\mathscr{S}}J$. Now $\Phi[x]$ is a principal ideal domain; so, by 2.1, $I^* \neq J^*$ and $I^{**} \neq J^{**}$. Hence if for some left ideal I, I were not equal to I^{**}, we could take $I \subset J \subset I^{**}$ and obtain a contradiction. Therefore $I = I^{**}$ for all nonzero left ideals I. Hence I can be embedded in a free module G so that G/I is isomorphic to a submodule of a free module. By 2.2, gl dim $R \leqslant 2$; so h.d. $G/I \leqslant 1$, and then h.d. $I = 0$. Therefore, every left ideal of R is projective, and R is hereditary.

THEOREM 2.4. *Let F be a field of characteristic zero, D a division F-algebra, $R = D \otimes A_1$. Then* gl dim $R \leqslant 2$, *with equality if and only if there is an isomorphism $A_1 \to D_n$ for some n.*

Proof. By 2.2 and 2.3, we have only to show that if there is an isomorphism $A_1 \to D_n$, then R is not hereditary. It is enough to show that the matrix ring R_n is not hereditary. By hypothesis, there are elements e, f in R_n, commuting with x and y, such that $ef - fe = 1$. Hence $(x - f)(y - e) = (y - e)(x - f)$, so $(x - f) R_n \cap (y - e) R_n \supseteq (x - f)(y - e) R_n$. If $(y - e)\alpha \in (x - f) R_n$, then $\alpha \in D_n[y] + (x - f) R_n$. If $(y - e)\alpha \in (x - f) R_n$ with α actually in $D_n[y]$, then clearly $\alpha = 0$. Hence $(x - f) R_n \cap (y - e) R_n$ is equal to $(x - f)(y - e) R_n$. Thus if $I = R_n(x - f) + R_n(y - e)$, we have $I^* = (x - f)^{-1}R_n \cap (y - e)^{-1}R_n = R_n$. However, I is a proper left ideal in R_n; so if I were projective, we would have $I = I^{**}$, $I^* \neq R_n$. Hence I is not projective, so R_n is not hereditary.

Remark. The idea of Krull dimension for a noncommutative ring was introduced by Gabriel [4]. Let R be as in 2.4. We have seen that for a left ideal I in R, I^{**}/I has finite length, and that $I = I^{**}$ for all I if and only if R is hereditary. Thus by Lemma 3.1 of [1], the Krull dimension $k(R)$ of R is $\leqslant$ gl dim R.

3. In this section, we suppose that F is an algebraically closed field, A an F-algebra. We suppose either that A is a filtered algebra whose associated graded ring is a finitely generated commutative F-algebra, or else that the cardinal of F is greater than the dimension of A. This assumption ensures that endomorphisms of simple A-modules are just scalar multiplications (see [3]).

THEOREM 3.1. *Let B be any F-algebra, and put $R = A \otimes B$. Let M be a maximal left ideal of A, I a left ideal of R containing RM. Then $I = RN + RM$ for some left ideal N in B.*

Proof. First of all, if γ is an element of A which is not in $F + M$, then $M + M\gamma = A$. For otherwise, $M\gamma \subseteq M$, so that γ induces an endomorphism of A/M which is not a scalar multiplication. Let $N = I \cap B$ and suppose $\{v_i, w_j\}$ is an F-basis of B, where $\{w_j\}$ is a basis of N. Then every element of R can be expressed in the form $\sum (u_i v_i + t_j w_j)$ with u_i, t_j in A. Such an element belongs to $RM + RN$ if and only if all the u_i belong to M. Let α be an element of R not in $RM + RN$. We show that $R\alpha + RM + RN$ contains elements of B not in N. Without loss of generality, we can suppose $\alpha = \sum_1^n u_i v_i$. We proceed by induction on n. The assertion is clear if $n = 1$ or if $\alpha \in B$. Therefore, suppose $n > 1$ and $\alpha \notin B$. Then $u_i \notin F$ for some i, say $u_1 \notin F$. We may suppose $u_1 \notin F + M$ and then by the remark at the beginning of the proof, there exists $\tau \in M$ such that $\tau u_1 \equiv 1 \pmod{M}$. Thus

$$R\alpha + RM + RN \supseteq R(v_1 + \beta) + RM + RN,$$

where β has shorter length than α, and $v_1 + \beta \notin RM + RN$. If β belongs to $F + RM + RN$ or to B, we need go no further. Otherwise as before, there exists $\tau_1 \in M$ such that $\tau_1\beta \notin RM + RN$. Now $\tau_1\beta$ has the same length as β; so, by the induction hypothesis, $R\tau_1\beta + RM + RN$ contains elements of B not in N. Then so does $R\tau_1(v_1 + \beta) + RM + RN$; hence so does $R(v_1 + \beta) + RM + RN$; and finally so does $R\alpha + RM + RN$. By induction, this holds for all α not in $RM + RN$. Now clearly $I = RM + RN$.

COROLLARY 1. *With the notation of the theorem, if B is a division algebra, then RM is a maximal left ideal in R.*

COROLLARY 2. *Let L be a left ideal in A such that A/L has finite length. Then there is a series of left ideals between R and RL whose factors have the form $R/RN + RM$ for suitable left ideals N (in B) and maximal left ideals M (in A). If also B/N has finite length, $R/RN + RL$ has finite length, and has a composition series with factors of the form $R/RM' + RM$, where M, M' are maximal left ideals in A, B, respectively.*

Proof. Suppose $A = L_1 \supset L_2 \supset \cdots \supset L_t = L$ is a composition series between A and L. Then $L_i/L_{i+1} \simeq A/M$ for suitable maximal left ideals M in A. Hence there is a homomorphism $R/RM \to RL_i/RL_{i+1}$ onto RL_i/RL_{i+1}. Now apply the theorem. The last part follows by induction on the length of B/N.

THEOREM 3.2. *Suppose A is hereditary and Noetherian, and let D be a division F-algebra such that $R = D \otimes_F A$ is a Noetherian integral domain of finite global dimension. Let D' be the quotient division ring of A. Then R is hereditary if and only if $D' \otimes D$ is hereditary.*

Proof. If R is hereditary, $D' \otimes D$ is hereditary because it is a partial quotient ring of R. Suppose $D' \otimes D$ is hereditary. Let J, L be left ideals in R

with $J \supset L$, $J^* = L^*$ and J/L cyclic. Let $\mathscr{S}$ be the set of nonzero elements in A; then R has a quotient ring $R_{\mathscr{S}} = D' \otimes D$. Hence by 2.1, $J \subseteq D'L$ and J/L is a homomorphic image of R/RN for some nonzero left ideal N in A. As A is hereditary, A/N has finite length. By Corollary 2 above, R/RN has finite length and has a composition series for which all the factors have the form R/RM for suitable maximal left ideals M in A. Hence by the Jordan–Hölder theorem, J/L has a composition series with factors of this form. Therefore, for every left ideal I, I^{**}/I has finite length and (if nonzero) has a composition series with factors of this form. Now RM is R-projective, so a stepwise argument shows that h.d. $I \leqslant$ h.d. I^{**}. As in 2.3, if gl dim $R \leqslant n$, h.d. $I^{**} \leqslant n - 2$ (or zero). Hence h.d. $I \leqslant n - 2$, so gl dim $R \leqslant n - 1$. We can now repeat the argument and eventually we get gl dim $R \leqslant 1$.

Suppose F has characteristic zero, and suppose A is such that $D' \otimes A_1$ is hereditary (see 2.4). Then if D is the quotient division ring of A_1, Theorem 3.2 shows that $D \otimes A$, $D' \otimes D$ are hereditary.

Finally, we have

THEOREM 3.3. *Let A and B be hereditary Noetherian algebras satisfying the condition stated at the beginning of this section, and such that $R = A \otimes B$ is a Noetherian integral domain of finite global dimension. Let D be the quotient division ring of A, D' the quotient division ring of B, and suppose $D \otimes D'$ is hereditary. Then* gl dim $A \otimes B \leqslant 2$.

Proof. By 3.2, $A \otimes D'$, $D \otimes B$ are hereditary. Then as in the proof of 3.4, if I is a left ideal in R, I^{**}/I has a composition series in which every factor has (up to isomorphism) the form $R/RM + RM'$ for suitable maximal left ideals M, M' in A, B, respectively. Then using the same argument as in 3.4, it is enough to show that h.d. $(RM + RM') \leqslant 1$. In the exact sequence $0 \to RM \cap RM' \to RM \oplus RM' \to RM + RM' \to 0$, $RM \oplus RM'$ is projective; so we have to show that $RM \cap RM'$ is projective. However, $RM \cap RM' = MM' \cong M \otimes_F M'$. Since M is A-projective and M' is B-projective, it now follows that $M \otimes_F M'$ is $A \otimes B$-projective. This completes the proof.

COROLLARY. *If F has characteristic zero, if D is the quotient division ring of A, and there is no isomorphism $A_1 \to D_n$ for any n, then* gl dim $A \otimes A_1 \leqslant 2$.

Remark 1. As far as the corollary is concerned, the most obvious case in which the condition on A is not satisfied is $A = A_1$. One is then considering the simple algebra $A_2 = A_1 \otimes A_1$. Our method gives no new information about the global dimension of A_2 (its Krull dimension is known to be 2; see [4]).

Remark 2. As in the remark following Theorem 2.4, we have $k(R) \leqslant \text{gl dim } R$ for algebras $R = A \otimes B$ satisfying the conditions of Theorem 3.3.

We now give some examples of division algebras satisfying the condition in Theorem 2.3.

1. Any division F-algebra finite-dimensional over its centre.

2. Any division F-algebra whose centre Φ is uncountable and which is algebraic over Φ. For D_n is also algebraic over Φ (see [5, p. 247]). If there were an isomorphism $A_1 \to D_n$, the Φ-subspace generated by the image of A_1 in D_n would be finite-dimensional over Φ, which is impossible.

3. Let H be an ordered group, D the set of all formal series $\sum_1^\infty \alpha_i h_i$ with α_i in F, and h_i in H, $h_1 < h_2 < h_3 < \cdots$.

With addition and multiplication defined as usual, D becomes a division F-algebra. This construction is due to Neumann [5, p. 188]. One can show directly that any equation $AB - BA = I$ in D_n would imply an equation $\sum (A_i B_i - B_i A_i) = I$ in F_n. Now the trace of the left side is zero; so this equation is impossible over a field of characteristic zero. Thus there is no isomorphism $A_1 \to D_n$.

Note added in proof, January 1972. G. S. Rinehart has pointed out that 2.4 holds also in characteristic p, provided one replaces "isomorphism $A_1 \to D_n$" by "non-zero homomorphism $A_1 \to D_n$".

Acknowledgment

The author wishes to thank J. C. McConnell for many helpful conversations.

References

1. R. Hart, Krull dimension and global dimension of simple Ore-extensions, *Math. Z.* **121** (1971), 341–345.
2. I. Kaplansky, "Commutative Rings," Mathematics Notes, Queen Mary College, 1966.
3. D. Quillen, On the endomorphism ring of a simple module over an enveloping algebra, *Proc. Amer. Math. Soc.* **21** (1969), 171–172.
4. R. Rentschler and P. Gabriel, Sur la dimension des anneaux et ensembles ordonnés, *Compt. Rend. Acad. Sci. Paris* **265** (1967), 712–715.
5. N. Jacobson, "Structure of Rings," Amer. Math. Soc. Colloquium Publications, No. 37, Providence, R. I., 1956.